Computation of the Bounding Surface of a Dipole Field
in a Plasma by a Moment Technique

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Abstract. The cavity formed by a plasma-free dipole magnetic field in a field-free stationary plasma is determined by a numerical method. The shape of the cavity is found by requiring that each multipole moment of the surface currents just cancel the corresponding moment due to the source of the field. For a dipole source of moment $M$ in a plasma whose pressure is $p$, there is a cusp on the polar axis at 0.64 times the equatorial radius, which is $0.82615\ M^{1/3}p^{-1/6}$. The relative accuracy of Beard's approximate method of solution is examined and found to be poor. A way is given in which the moment method might be used to obtain the shape of the cavity produced in the solar wind by the earth's magnetic field.

1. Introduction. It has long been suggested [ Chapman and Ferraro, 1930; Ferraro, 1952] that there exists a flow of plasma from the sun which, because of its conductivity, compresses the earth's magnetic field, confining it to a cavity whose shape is yet to be determined. When the plasma first arrives, there will of course be a transient disturbance, but we shall assume that if the flow is steady this is followed by a steady state in which the geomagnetic field and the plasma occupy different regions of space. Later the plasma will diffuse into the magnetic field as a result of its finite conductivity and any instabilities there may be. However, we will consider here only the initial steady state without penetration and we will ignore the question of stability.

Since the problem was first suggested numerous attempts have been made to deduce the shape of the cavity, but no accurate solution has yet been obtained. However, similar, but simpler, problems may be investigated and the results obtained may have some geophysical significance in their own right. They also assist in developing techniques for an attack on the main problem and provide tests for the relative accuracy of various proposed approximation methods. For example, Dungey [1961] has given an exact solution of the case of a two dimensional dipole normal to a uniform wind. Another useful case is the three dimensional dipole surrounded by a stationary uniform pressure plasma. We treat this latter problem below, giving in section 2 the general moment technique in a form which it is hoped can be generalized to the problem of a dipole in a steady plasma flow, and in section 3 the specific solution. In section 4 the results of this method are compared with the results which are obtained by using an approximate boundary condition due to Beard [1960].

All of these problems involve two regions, separated by a surface at which an appropriate boundary condition applies. It is generally agreed [Beard, 1960] that in the geomagnetic situation this surface is actually a current sheath of the order of a kilometer think, and that this thickness can be neglected. The inner region contains only vacuum magnetic field with specified sources (in our case a dipole at the origin) though an extension to allow for the presence of plasma could presumably be devised. The outer region contains only plasma, and thus it is in effect a perfect diamagnetic. Once the surface is specified, it is a straightforward problem in electromagnetic theory to determine what the field must be inside the cavity using the boundary condition that all space outside the cavity is filled with a diamagnetic. With the surface specified, it should also be a straightforward problem, given the plasma motion and pressure at infinity, to determine the normal pressure p of the plasma at each point on the surface. Then the boundary condition which must be satisfied by the true (or self-consistent) surface [Ferraro, 1952] is that this plasma pressure must be balanced by the magnetic pressure of the field, $\utilde{B}$ , just inside the surface. In Gaussian units:

~ means bold-faced type

$$p = \frac{B^2}{8\pi} \tag{1}$$

In general, if the surface is not known, it is possible to obtain it only by making successive trials, testing each against condition (1) to determine its accuracy. There is always considerable difficulty, however, in determining just how to change the surface each time in order to reduce the discrepancy in (1). Also the straightforward problem in electromagnetic theory of determining the field inside the cavity from the diamagnetic boundary condition actually involves a lengthly numerical solution of a partial differential equation throughout the cavity.

This partial differential equation can be avoided and the problem can be greatly simplified by making use of the surface currents, $\utilde{J}$ , on the boundary. Since the magnetic field is zero outside, a well known boundary condition prescribes that J must be just $B/4\pi$ and that $\utilde{J}$ is perpendicular to $\utilde{B}$ . Using this condition, (1) becomes

$$p = 2\pi J^2 \tag{2}$$

Again starting with a trial surface, p is calculated at each point of the surface from conditions in the plasma. This is trivial for the case of the stationary uniform pressure plasma considered here. For the zero temperature solar wind problem to be considered subsequently, we shall assume elastic reflection of noninteracting particles, in which case p varies as $\cos^2\psi$ where $\psi$ is the angle between the inward normal to the surface and the wind direction. With p determined, equation (2) determines the magnitude of J at each point on the surface. One then uses symmetry and the equation of continuity to determine, if possible, the vector surface current everywhere. (If this fails, an alternative approach is to reverse these two steps, specifying a trial current distribution and constructing the surface so that (2) is satisfied.) Knowing the surface current, it is then a simple matter to calculate from a surface integral the magnetic field anywhere to determine whether it satisfies the boundary conditions. Thus if just inside the surface the field is everywhere tangent to the surface, has the magnitude $4\pi J$ and is perpendicular to $\underset{\sim}{J}$ , then the boundary condition is satisfied. Dungey [1961] in his two dimensional solution used such a boundary condition, but was able to use complex variable theory and got the exact magnetic field and boundary by a conformal transformation. Beard [1960] obtains a differential equation for an approximation to the surface by using the following approximate boundary condition. He requires that at each point on the surface the tangential component of the unperturbed field is just half of the discontinuity, $4\pi J = (8\pi p)^{\frac{1}{2}}$, required by the surface current deduced from the plasma pressure. Thus he assumes that the tangential component of the field produced by all the surface current is precisely equal to that due to the sources, and that the normal components due to the two causes are precisely equal and opposite. This assumption is only roughly true and his result is therefore quite approximate. Ferraro [1960] uses essentially the same approximate boundary condition to get a solution of the case of a straight wire carrying a current in a uniform plasma wind.

2. The moment technique. Actually it is very difficult to compute numerically the value of the magnetic field near the surface because the integrand of the integral used diverges as the field point considered approaches the surface. However, in practice it is not necessary to consider the field just inside the surface, for a convenient equivalent condition is available. It is that the magnetic field be zero everywhere outside the surface, and a condition equivalent to this is that all the multipole moments of the source be zero. Thus the procedure used is to assume a functional form for the surface with a large number of undetermined parameters. A series of sets of the parameters are chosen, each set defining a trial surface. For each surface the current distribution and the multipole moments of this current distribution are calculated. These moments are added to those of the fixed sources and the parameters are so adjusted that all the resultant multipole moments are zero. A convenient way to do this for a fixed dipole source is to calculate the rate of change of each of the moments with respect to each of the parameters. With this information a set of linear equations is constructed and solved to determine how each of the parameters should be changed in order to reduce the magnitude of all the moments due to

the surface currents except the dipole moment. This process is iterated until a surface is obtained whose surface currents have only a dipole moment. If the surface is then scaled so that this dipole is equal and opposite to the earth's dipole, then the solution is obtained. Of course with only a finite number of parameters only a finite number of moments can be reduced to zero, but only the lower moments contribute much to the field. If one requires the surface to be particularly accurate in certain regions, one can consider fewer moments and also require the field to be zero at selected points near these regions. Again, these points cannot be too near, for reasons of convergence.

3. Uniform pressure case. Consider a magnetic dipole surrounded by a stationary plasma of uniform pressure, and let the standard spherical polar coordinates be

$$(r, \theta, \phi) = (\rho r_o, \frac{\pi}{2} - \alpha, \phi) \tag{3}$$

where $\alpha$ is the usual magnetic latitude, $\rho$ is a dimensinless length and $r_o$ is a convenient unit of length, chosen to be the radius in the equational plane to the point where the magnetic pressure of the undisturbed dipole field equals the gas pressure. Thus

$$r_o^3 = M (8 \pi p)^{-\frac{1}{2}} \tag{4}$$

where M is the moment of the dipole. Clearly the bounding surface has axial symmetry and the surface current is $\underset{\sim}{J} = (p/2 \pi)^{\frac{1}{2}} \underset{\sim}{e}_{\phi}$, having the same magnitude everywhere on the surface. This is the main simplification of this case over the case of the wind, where both the magnitude and direction of the current vary in a difficult manner over the surface.

Let $\underset{\sim}{r}_2$ be a point outside the surface, where we require $\underset{\sim}{A}(r_2)$, the vector potential of the magnetic field, and let $\underset{\sim}{r}$ be a point on the surface. Then in Gaussian units for a general surface

$$\underset{\sim}{A}(\underset{\sim}{r}_2) = \int \frac{\underset{\sim}{J}(\underset{\sim}{r})\, dS}{|\underset{\sim}{r}_2 - \underset{\sim}{r}|} \tag{5}$$

For our symmetrical case we may expand in Legendre polynomials and integrate over $\phi$ to obtain

$$\underset{\sim}{A}(\underset{\sim}{r}_2) = \underset{\sim}{e}_\phi \sum_{n=1}^{\infty} (2\pi p)^{\frac{1}{2}} \frac{r_o^{\,n+2}}{n(n+1)} \frac{P_n^1(\sin\alpha_2)}{r_2^{\,n+1}} I_n \tag{6}$$

where

$$I_n = \int_{-\pi/2}^{\pi/2} \rho^{n+1} \left[\rho^2 + \left(\frac{d\rho}{d\alpha}\right)^2\right]^{\frac{1}{2}} P_n^1(\sin\alpha)\cos\alpha\, d\alpha \tag{7}$$

The $I_n$ are proportional to the $2^n$-pole moments of the surface current, so we now require that $\rho(\alpha)$ be so chosen that all the $I_n$ except $I_1$ are zero and that the dipole moment of the surface currents just cancels that of the source at the origin. Since the vector potential of a dipole of strength $M$ is $\underset{\sim}{e}_\phi M r_2^{-2} P_1^1(\sin\alpha_2)$, comparison with (6) and elimination of $r_o$ by means of (4) gives

$$I_1 = 4 \tag{8}$$

The computation was carried out for a trial function of the form

$$p = C\left[1 - \sum_{s=1}^{N} c_s\, \alpha^{2s}\right] \tag{9}$$

At first we set $C = 1$ and put no restriction on the value of $I_1$. Symmetry again guarantees that $I_n = 0$ for $n$ even, so we need to consider $I_n$ only for $n = 3,5,7, \ldots 2N+1$. It is an easy matter to differentiate the $I_n$ under the integral sign and obtain analytic expressions for the rates of change of the $I_n$ with respect to the various $c_s$. Hence the Generalized Newton's Method was used to determine the $c_s$ which reduced the $I_n$ to zero. Finally $I_1$ is made equal to 4 by adjusting C, which is seen to be the equatorial radius. The computation was carried out on a Burroughs 220 computer for various values of $N$ up to seven. For the case $N = 7$ the numerical results are given in Table 1 and the resulting cross section is plotted in Figure 1.

TABLE 1.

Coefficients in the Equation for the Surface

$$
\begin{aligned}
C &= 1.41395 \\
c_1 &= 0.120039 \\
c_2 &= 0.004180 \\
c_3 &= 0.001085 \\
c_4 &= -0.000200 \\
c_5 &= 0.000597 \\
c_6 &= -0.000326 \\
c_7 &= 0.000094
\end{aligned}
$$

The radius in the equatorial plane is $1.41395\ r_0$, the intersection on the dipole axis is $0.899\ r_0$, and the maximum height is $1.073\ r_0$. The equatorial radius is curiously close to $\sqrt{2}\ r_o$. If this should be exact, it might indicate that the true surface could be given by an unexpectly simple expression which could presumably be derived, but this has not been pursued.

It is true that at the pole the last few terms of equation (9) are of the order of 7% of the first term but this does not indicate an error of that order there. The coefficients in Table 1 are not the first seven terms in the power series expansion of the true surface. They are the coefficients of the polynomial of degree fourteen which most closely approximates the true surface. We have two reasons for believing that the solution is very accurate even near the pole. First, when the computation was carried out with only four parameters, the radius of the computed surface near $\alpha = 90^o$, where agreement was worst, was only about one percent greater than the corresponding radius of the seven parameter surface. Second, when $c_1$ was changed so as to decrease the radius to the surface by only 0.1% at the pole, the residual fields at distances greater than $0.3r_o$ outside the surface (calculated as described in the test of the next section) were increased by a factor of ten or more. A major feature of interest in this computation, in addition to providing a test of the moment technique, is that it indicates that the surface very definitely has cusps at the poles and that these cusps do not go clear to the origin as has been suggested, but rather interesect the axis at a finite distance. The cusps undoubtedly interesect the axis tangentially in reality, but such a surface could not be represented by a polynomial with a finite number of terms such as we have used. It is of interest to note, however, that the more parameters we used the steeper the angle of intersection was. It is easy to see that these are the results that should be expected. Consider a cavity in a medium of zero permeability. If there were a finite angle between the surface and the axis, the field there would be zero, and if the cusp were at the dipole the field would be infinite; either condition is inconsistent with equation (1).

If we define the field just inside the surface to be $B_s = (8\pi p)^{\frac{1}{2}}$, then it is a simple matter to see that the change in the field, $\Delta B_o$ at the origin due to the surface currents is

$$\Delta B_o = B_s \int_0^{\frac{\pi}{2}} \cos^2\alpha \, \left(1 + \left(\frac{dr}{r d\alpha}\right)^2\right)^{\frac{1}{2}} \, d\alpha \qquad (10)$$

For a sphere the integral is just $\pi/4$, and for any other surface it would be slightly greater. For our surface it is 0.76933. Thus a 10 $\gamma$ disturbance in the geomagnetic field at the earth could arise from a sudden change of pressure of $2.52 \times 10^{-10}$ dynes/cm$^2$ on the surface (i.e. a particle density times temperature of $1.83 \times 10^6$ K$^o$/cm$^3$ or a kinetic energy density of $1.58 \times 10^2$ ev/cm$^3$ ).

4. Comparison with other results. We have felt it desirable to devise a technique for checking the above solution that is essentially independent of the method by which it was obtained. Such a check would provide a better estimate of the accuracy of the solution than would an examination of the residual moments and of the roundoff and similar errors in the computation, and at the same time it would show up any coding errors. Of even greater importance, however, such a check could be applied to

solutions obtained by other methods and thus afford an objective way of estimating their relative accuracy. The moment technique forces the field to go to zero to a very high order at large distances from the surface. Other procedures which may be devised usually attempt to force the field to zero just outside the surface. Of course with the true solution the field would be exactly zero everywhere outside the surface. Thus it seems that a very reasonable test of any proposed surface would be to calculate the field outside that surface at different distances from the surface and in different directions. To be specific, we calculated (at various radii along the polar axis and in the equatorial plane) the field due to the surface, subtracted this field from the field of the dipole located at the origin, and then divided the result by the dipole field. This gives a number which would be zero everywhere outside the surface for the true surface and would be one everywhere if the surface were removed altogether. For our solution, where the two dipole moments are the same, this test will clearly give zero at large distances so the real test will come at small distances. The real test of a surface derived by a process which minimizes the field at the surface may come at greater radii. The computations for this test were carried out on a Burroughs 220 computer, replacing the surface by ninety-eight current loops. The results of this test for the moment surface are given in Table 2. The values on the polar axis may be incorrect by as much as 5% due to truncation error. The truncation error was removed from the equatorial values by subtracting the solution for a sphere with a $\cos\alpha$ current variation, which should theoretically be zero everywhere and which therefore equals the truncation error in practice. Since the surface approximates a sphere near the equator and the $\cos\alpha$ current approximates a uniform current near the equator, the truncation error must be very nearly the same for both cases near the surface at the equator. The inherent roundoff error in the calculation was about $.2 \times 10^{-5}$.

TABLE 2.— Ratio of Net Field to Dipole Field $\times 10^5$

| Distance from the surface— Fraction of Equatorial Radius | Moment Surface | | Beard Surface | |
|---|---|---|---|---|
| | On the Polar Axis | In the Equatorial Plane | On the Polar Axis | In the Equatorial Plane |
| 0.04 | -905 | -0.4 | -61078 | 7126 |
| 0.08 | -222 | +0.2 | -42966 | 6721 |
| 0.16 | - 23 | 0.6 | -27676 | 5997 |
| 0.32 | - 2.7 | 0.5 | -15913 | 4844 |
| 0.64 | - 0.9 | 0.5 | - 7947 | 3324 |
| 1.28 | - 0.2 | 0.2 | - 3378 | 1317 |
| 2.56 | - 0.1 | 0.3 | - 1222 | 773 |
| 5.12 | - 0.0 | 0.5 | - 336 | 267 |
| 10.24 | 0.2 | 0.0 | - 110 | 81 |

As a result for comparison with the moment technique, the uniform pressure problem was also solved by Beard's differential equation technique. To get the equation for $r(\alpha) = \rho(\alpha) r_o$, set the magnetic pressure of the tangential component of a field $1/f$ times as strong as the earth's field equal to the plasma pressure

$$(1/8\pi)(f^{-1}\underset{\sim}{e}_t \cdot \underset{\sim}{B}(r, \alpha))^2 = p \qquad (11)$$

or in full: $$\left[(\underset{\sim}{e}_\alpha + \frac{d\rho}{\rho d\alpha}\underset{\sim}{e}_r).(\cos\alpha\underset{\sim}{e}_\alpha - 2\sin\alpha\,\underset{\sim}{e}_r)\frac{M}{\rho^3 r_o^3}\right]^2 = 8\pi f^2 p\left[1 + (\frac{d\rho}{\rho d\alpha})^2\right] \qquad (12)$$

Call $\rho(0) = \rho_e$ and note that $d\rho/d\alpha = 0$ at $\alpha = 0$ by symmetry. Inserting these values, and the value of $r_o$ from (4), into (12) one obtains the relation

$$f\rho_e^3 = 1 \qquad (13)$$

and the differential equation

$$\left[\cos\alpha - 2\sin\alpha\,\frac{d\rho}{\rho d\alpha}\right]^2 = (\frac{\rho}{\rho_e})^6\left[1 + (\frac{d\rho}{\rho d\alpha})^2\right] \qquad (14)$$

When equation (14) is solved it gives $\rho(\alpha)/\rho_e$. Then $\rho_e$ is determined by the condition that $I_1 = 4$. Since equation (14) is of second degree there are two such solutions. The appropriate solution is plotted in Figure 1 and it is seen that it differs significantly from the moment technique result near the pole. The result of the test applied to this surface is also given in Table 2. Clearly the moment technique gives a net field outside which is about one thousandth of that given by the surface derived using Beard's boundary condition.

There is also an interesting sidelight that can be gleaned from these calculations. There has been some discussion recently as to whether the factor f which Beard assumes to be ½ should not be closer to 1/3. From equation (13) we see that in this three dimensional case

$$f = \rho_e^{-3} = (1.39577)^{-3} = 0.36775 \qquad (15)$$

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Since completing the above solutions, we have learned that Dr. Ralph J. Slutz [see adjacent article] has been working on the same problem with a quite different approach. He has kindly provided us with information on his solution which lies about 0.04 $r_o$ inside our surface at the equator, agrees with our surface to within the accuracy of the data supplied us between latitudes of 20° and 85° and lies about .01 $r_o$ outside our surface just at the pole. Thus his surface disagrees with ours only in that it is more cylindrical near the equator. His method of calculation approximated the surface by many small flat triangles rather than a smooth function, so it has not been clear how to apply our test to it fairly. However, we did test a smooth surface which passed through the vertices of his triangles and concluded from the residual fields in the equatorial plane that our surface is correct in the equatorial ~~plane~~ region to within a few tenths of a percent. The essential coincidence, except for a three percent discrepancy near the equator, of the surfaces obtained by such different numerical methods gives considerable confidence in both.

Acknowledgements. The completion of this research has been supported by a grant from the National Aeronautics and Space Administration. One of us (J.E.M.) is also grateful to the National Science Foundation for Fellowships during the time he has worked on this research.

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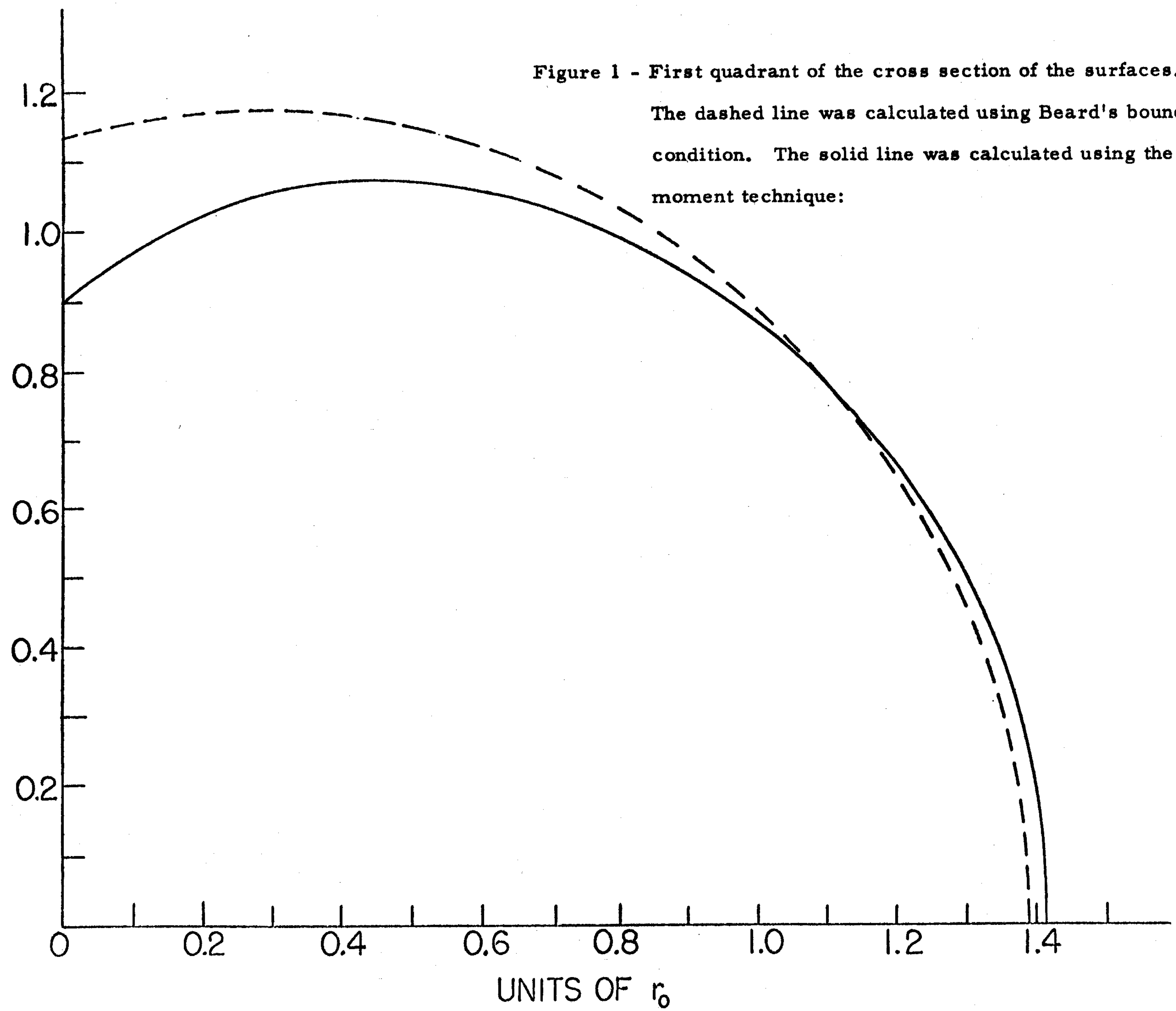


**Figure 1 - First quadrant of the cross section of the surfaces. The dashed line was calculated using Beard's boundary condition. The solid line was calculated using the moment technique:**